

Cohomological vanishing on Siegel modular varieties and applications to lifting Siegel modular forms*

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Abstract

We use vanishing results for sheaf cohomology on Siegel modular varieties to study two lifting problems:

- (a) When can Siegel modular forms (mod p) be lifted to characteristic zero? This uses and extends previous results for *cusp* forms by Stroh and Lan-Suh.
- (b) When is the restriction of Siegel modular forms to the boundary of the moduli space a surjective map? We investigate this question in arbitrary characteristic, generalising analytic results of Weissauer and Arakawa.

1 Introduction

A venerable technique in arithmetic geometry is to take an object defined over the integers and study its reductions modulo various primes. In the case of classical modular forms with algebraic integer coefficients, this reduction process gives rise to *Serre-type modular forms (mod p)*, whose q -expansion coefficients are in $\overline{\mathbb{F}}_p$. There is a more intrinsic way of producing modular forms with coefficients in $\overline{\mathbb{F}}_p$: in the moduli-theoretic definition of modular forms, consider the moduli space of elliptic curves over $\overline{\mathbb{F}}_p$. This gives rise to *Katz-type modular forms (mod p)*. The natural question is whether the two definitions agree—we formulate this question as follows: “Do all (Katz-type) modular forms (mod p) lift to characteristic zero?”

[12, Theorem 1.7.1] provides a partial positive answer:

Theorem 1.1 (Katz). *All modular forms (mod p) of weight $k \geq 2$ and level $\Gamma(N)$ with $N \geq 3$ lift to characteristic zero.*

(We give a variant of Katz’s argument in the proof of Theorem 5.6.)

In the context of his computational exploration of Serre’s conjecture, Mestre found examples of modular forms (mod 2) of weight 1 that do not lift to characteristic zero. His smallest example appears in level 1429. These computations were reproduced and extended by Wiese; both his and Mestre’s approach appear as appendices to [5]. The

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interested reader would also benefit from reading Buzzard's note on computing weight 1 forms [3], as well as the novel and systematic approach given in Schaeffer's PhD thesis [20].

The question of liftability of modular forms (mod p) on *higher rank groups* has recently received some attention. Stroh proved that scalar-valued Siegel cusp forms (mod p) of degree 2 and weight $k \geq 4$ for $p > 2$, or degree 3 and weight $k \geq 5$ for $p > 5$, can be lifted to characteristic zero [21, Théorème 1.1]. More recently, Lan and Suh proved that on a Shimura variety X of PEL type, any cusp form (mod p) for $p \geq \dim(X)$ of strictly positive parallel cohomological weight lifts to characteristic zero [14, Theorem 4.1]. Restricted to the Siegel case, this gives liftability of scalar-valued Siegel cusp forms of degree g and weight $k \geq g + 2$ for $p \geq g(g + 1)/2$.

Our main results are Theorem 5.6 and Corollary 5.7, which extend these liftability theorems from Siegel *cusp* forms to Siegel *modular* forms. The arguments of Stroh and Lan-Suh are based on vanishing theorems for the cohomology of line bundles of cusp forms on a *toroidal compactification* of the Siegel modular variety. It is then necessary to investigate what happens along the boundary. Our strategy is to pass to the *Satake compactification*, whose boundary consists of strata isomorphic to Siegel modular varieties of *smaller degree*. In other words, these correspond to smaller instances of the problem, enabling us to set up an inductive argument. The missing ingredient is a comparison between higher cohomology of line bundles on the toroidal and Satake compactifications. This is our Theorem 5.4, which uses the vanishing of relative cohomology of cuspidal forms, proved recently and independently by Stroh [22] and Andreatta-Iovita-Pilloni [1].

A pleasant side effect of our strategy is that it also yields information on the surjectivity of the Siegel Φ -operator, which restricts Siegel modular forms to the boundary of the moduli space. We give these results in Section 7; in characteristic zero, we can even handle vector-valued forms, via a vanishing theorem for vector bundles on Siegel modular varieties described in Section 6.

There are no known examples of Siegel modular forms (mod p) of small weight that do not lift to characteristic zero. The naive search for such forms would require computing with Siegel modular forms of high level; however, this appears to be presently out of reach even if we restrict to the simplest setting of scalar-valued forms of degree 2.

Note that we assume $N \geq 3$ for most of the paper. In Section 8 we describe how to extend our results to the low level cases $N = 1$ and $N = 2$.

2 Siegel modular varieties and forms

Let $\mathcal{A}_{g,N}$ denote the moduli space¹ of principally polarized g -dimensional abelian varieties with full level N structure. It is a smooth quasi-projective scheme of dimension $g(g + 1)/2$ over $\mathbb{Z}[1/N]$, see [19, Theorem 7.9]. Let A denote the universal abelian variety, so that $f: A \rightarrow \mathcal{A}_{g,N}$ is a smooth morphism of relative dimension g .

The *Hodge bundle* \mathbb{E} is the rank g vector bundle on $\mathcal{A}_{g,N}$ defined by

$$\mathbb{E} = f_* \Omega_{A/\mathcal{A}_{g,N}}^1.$$

Let ρ be an irreducible representation of the algebraic group GL_g and let $\lambda = (\lambda_1, \dots, \lambda_g)$ be its highest weight vector. By applying ρ to the transition functions of the vector bundle \mathbb{E} , we obtain a rank $d = \dim \rho$ vector bundle \mathbb{E}^ρ . An important special case is $\rho = \det$, and we denote the resulting line bundle by $\omega = \mathbb{E}^{\det} = \det \mathbb{E}$.

¹Recall that, by assumption, $N \geq 3$.

Given a $\mathbb{Z}[1/N]$ -algebra B , the space of *Siegel modular forms* of degree g , weight ρ and level $\Gamma(N)$ with coefficients in B is the B -module defined by

$$M_\rho(N; B) = H^0(\mathcal{A}_{g,N}, \mathbb{E}^\rho \otimes B).$$

In particular, if p is a prime number not dividing N , the elements of $M_\rho(N; \overline{\mathbb{F}}_p)$ are called *Siegel modular forms (mod p)*.

A construction due to Ash-Mumford-Rapoport-Tai associates to a choice of combinatorial data (a cone decomposition) a *toroidal compactification* $\mathcal{A}_{g,N}^{\text{tor}}$ of $\mathcal{A}_{g,N}$. It is possible to choose $\mathcal{A}_{g,N}^{\text{tor}}$ in such a way that it is a smooth projective scheme over $\mathbb{Z}[1/N]$, containing $\mathcal{A}_{g,N}$ as a dense open subscheme, and such that the boundary divisor $\mathcal{A}_{g,N}^{\text{tor}} - \mathcal{A}_{g,N}$ is simple with normal crossings. Moreover, $\mathcal{A}_{g,N}^{\text{tor}} \times \text{Spec } \mathbb{C}$ is a smooth projective complex manifold when $N \geq 3$. There is a canonical extension of the Hodge bundle \mathbb{E} to a rank g vector bundle \mathbb{E}_{tor} on $\mathcal{A}_{g,N}^{\text{tor}}$. The line bundle $\omega_{\text{tor}} = \det \mathbb{E}_{\text{tor}}$ is the canonical extension of ω to $\mathcal{A}_{g,N}^{\text{tor}}$.

The *Satake compactification* $\mathcal{A}_{g,N}^{\text{Sat}}$ is the normal, proper scheme over $\mathbb{Z}[1/N]$ given by

$$\mathcal{A}_{g,N}^{\text{Sat}} = \text{Proj} \left(\bigoplus_{k \geq 0} H^0(\mathcal{A}_{g,N}^{\text{tor}}, \omega_{\text{tor}}^{\otimes k}) \right).$$

The main properties of $\mathcal{A}_{g,N}^{\text{Sat}}$ are given in [6, Theorem V.2.5]. It contains $\mathcal{A}_{g,N}$ as a dense open subscheme, and the line bundle ω on $\mathcal{A}_{g,N}$ extends to an ample line bundle ω_{Sat} on $\mathcal{A}_{g,N}^{\text{Sat}}$. There is a canonical extension of the Hodge bundle \mathbb{E} to a coherent sheaf (but not a vector bundle) \mathbb{E}_{Sat} on $\mathcal{A}_{g,N}^{\text{Sat}}$; similarly, any twist of the Hodge bundle \mathbb{E}^ρ on $\mathcal{A}_{g,N}$ extends canonically to a coherent sheaf $\mathbb{E}_{\text{Sat}}^\rho$ on $\mathcal{A}_{g,N}^{\text{Sat}}$.

The Kocher principle states that, if $g > 1$, a Siegel modular form with coefficients in some $\mathbb{Z}[1/N]$ -algebra B extends uniquely to the Satake and toroidal compactifications:

$$\begin{aligned} H^0(\mathcal{A}_{g,N}^{\text{Sat}}, \mathbb{E}_{\text{Sat}}^\rho \otimes B) &= H^0(\mathcal{A}_{g,N}, \mathbb{E}^\rho \otimes B); \\ H^0(\mathcal{A}_{g,N}^{\text{tor}}, \mathbb{E}_{\text{tor}}^\rho \otimes B) &= H^0(\mathcal{A}_{g,N}, \mathbb{E}^\rho \otimes B). \end{aligned}$$

The Satake case can be found in [7, Proposition 5], and the toroidal case in [6, Proposition V.1.8].

If $D = \mathcal{A}_{g,N}^{\text{tor}} - \mathcal{A}_{g,N}$ denotes the boundary divisor of a toroidal compactification $\mathcal{A}_{g,N}^{\text{tor}}$, we define the space of *Siegel cusp forms* of weight ρ and level $\Gamma(N)$ with coefficients in B to be

$$S_\rho(N; B) = H^0(\mathcal{A}_{g,N}^{\text{tor}}, \mathbb{E}_{\text{tor}}^\rho(-D) \otimes B).$$

In other words, these are the Siegel modular forms that vanish along the boundary of $\mathcal{A}_{g,N}^{\text{tor}}$. Their definition is independent of the choice of toroidal compactification [6, page 144]. It is also possible to define cusp forms using the Satake compactification. If $\Delta = \mathcal{A}_{g,N}^{\text{Sat}} - \mathcal{A}_{g,N}$ denotes the boundary of the Satake compactification and \mathcal{S}_ρ is the sheaf kernel of the restriction $\mathbb{E}_{\text{Sat}}^\rho \rightarrow \mathbb{E}_{\text{Sat}}^\rho|_\Delta$. By [7, Proposition 7], the global sections of \mathcal{S}_ρ are precisely the cusp forms of weight ρ and we have

$$S_\rho(N; B) = H^0(\mathcal{A}_{g,N}^{\text{Sat}}, \mathcal{S}_\rho \otimes B).$$

3 Review of cohomology and base change

Siegel modular forms are global sections of certain vector bundles. The issue of their liftability from positive characteristic to characteristic zero is thus a question of certain cohomology groups commuting with base change. This is an essential topic in algebraic geometry, treated in many of the standard references. However, to our knowledge, none of these references contains a precise statement of the result we need (Corollary 3.2). We give a proof in this section, which does little more than piece together the necessary ingredients from [9, Chapter 7] and [11, Chapter III]. Another useful treatment of these questions can be found in [23, Chapter 28].

Theorem 3.1 ([9, Corollary 7.5.5]). *Let A be a local noetherian ring with residue field $k = A/\mathfrak{m}$. Let T_\bullet be a homological functor $A\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ that commutes with direct limits. Suppose that for every q and every finitely generated A -module M , $T_q(M)$ is finitely generated and the canonical homomorphism*

$$\widehat{T_q(M)} \longrightarrow \varprojlim_n T_q(M \otimes_A A/\mathfrak{m}^{n+1})$$

is bijective.

(i) *If $T_q(k) = 0$ then $T_q(M) = 0$ for any A -module M , T_{q+1} is right exact and T_{q-1} is left exact.*

(ii) *If $T_{q-1}(k) = T_{q+1}(k) = 0$ then T_q is exact, the canonical homomorphism*

$$T_q(A) \otimes_A M \longrightarrow T_q(M)$$

is bijective and $T_q(A)$ is a free A -module.

Corollary 3.2. *Let A be a local noetherian ring with residue field $k = A/\mathfrak{m}$. Let X be a projective scheme over $\text{Spec } A$ and \mathcal{F} a coherent \mathcal{O}_X -module, flat over $\text{Spec } A$.*

(i) *If $H^q(X \times \text{Spec } k, \mathcal{F}) = 0$, then $H^q(X, \mathcal{F} \otimes_A M) = 0$ for any A -module M .*

(ii) *If*

$$H^{q-1}(X \times \text{Spec } k, \mathcal{F}) = H^{q+1}(X \times \text{Spec } k, \mathcal{F}) = 0,$$

then the canonical map

$$H^q(X, \mathcal{F}) \otimes_A M \longrightarrow H^q(X, \mathcal{F} \otimes_A M)$$

is bijective and $H^q(X, \mathcal{F})$ is a free A -module.

Proof. It suffices to show that $T_q(M) := H^q(X, \mathcal{F} \otimes_A M)$ satisfies the conditions of Theorem 3.1.

By [11, Proposition III.12.1], T_\bullet is a homological functor. It commutes with direct limits by [11, Proposition III.2.9].

By [11, Proposition III.12.2], there exists a complex L^\bullet of finitely generated free A -modules and an isomorphism of functors (in M)

$$T_q(M) \cong H^q(L^\bullet \otimes_A M).$$

In particular, the A -modules L^j are flat, so we are in the setting of [9, Section 7.4]. As the L^j are also finitely generated, [9, Proposition 7.4.7] indicates that $T_q(M)$ is finitely generated for any finitely generated M , and that the canonical map

$$\widehat{T_q(M)} \longrightarrow \varprojlim_n T_q(M \otimes_A A/\mathfrak{m}^{n+1})$$

is an isomorphism. □

4 Positivity of Hodge line bundles

We fix a choice of smooth, projective toroidal compactification $\mathcal{A}_{g,N}^{\text{tor}}$ of $\mathcal{A}_{g,N}$, we let D denote the simple normal crossings divisor $\mathcal{A}_{g,N}^{\text{tor}} - \mathcal{A}_{g,N}$, and we let \mathbb{E}_{tor} denote the canonical extension to $\mathcal{A}_{g,N}^{\text{tor}}$ of the Hodge bundle \mathbb{E} on $\mathcal{A}_{g,N}$. The objective of this section is to collect results about the positivity properties of $\omega_{\text{tor}} = \det \mathbb{E}_{\text{tor}}$ and deduce the vanishing of certain cohomology groups.

A line bundle \mathcal{L} on a projective variety X is *ample* if

$$(\mathcal{L} \cdot C)_X > 0$$

for every closed reduced irreducible curve $C \subset X$; here the intersection number $(\mathcal{L} \cdot C)_X$ is the coefficient of m in the polynomial $\chi(\mathcal{O}_C \otimes \mathcal{L}^{\otimes m})$, where $\chi(\mathcal{F})$ denotes the Euler characteristic of the sheaf \mathcal{F} . Our interest in ampleness is motivated by the following classical result (see [15, Theorem 4.2.1]):

Theorem 4.1 (Kodaira Vanishing Theorem). *Let X be a smooth complex projective variety and \mathcal{L} an ample line bundle on X . For all $q > 0$ we have*

$$H^q(X, \omega_X \otimes \mathcal{L}) = 0,$$

where ω_X is the canonical sheaf of X .

Unfortunately, ω_{tor} is generally not an ample line bundle. Luckily, it comes close enough that we can still deduce vanishing of cohomology, as we now see.

A line bundle \mathcal{L} on a projective variety X is *numerically effective* (nef) if

$$(\mathcal{L} \cdot C)_X \geq 0$$

for every closed reduced irreducible curve $C \subset X$.

A line bundle \mathcal{L} on an n -dimensional projective variety X is *big* if there is a constant $c > 0$ such that

$$\dim H^0(X, \mathcal{L}^{\otimes m}) \geq c \cdot m^n$$

for sufficiently large $m \in \mathbb{N}$.

For the purposes of vanishing of higher cohomology, we can replace *ample* with *nef and big* (see [15, Theorem 4.3.1]):

Theorem 4.2 (Kawamata-Viehweg Vanishing Theorem). *Let X be a smooth complex projective variety and \mathcal{L} a nef and big line bundle on X . For all $q > 0$ we have*

$$H^q(X, \omega_X \otimes \mathcal{L}) = 0,$$

where ω_X is the canonical sheaf of X .

Proposition 4.3. *The line bundle ω_{tor} on $\mathcal{A}_{g,N}^{\text{tor}}$ is nef and big.*

Proof. The line bundle ω_{Sat} on $\mathcal{A}_{g,N}^{\text{Sat}}$ is ample [6, Theorem V.2.5(1)]. There is a canonical morphism [6, Theorem V.2.5(2)]

$$\pi: \mathcal{A}_{g,N}^{\text{tor}} \longrightarrow \mathcal{A}_{g,N}^{\text{Sat}}$$

obtained as the normalization of the blow-up of $\mathcal{A}_{g,N}^{\text{Sat}}$ along a certain ideal sheaf [6, Theorem V.5.8]. Therefore π is surjective, proper and birational.

Since ω_{Sat} is ample, it is nef and big. Since pullbacks of nef line bundles along proper morphisms are nef [15, Example 1.4.4(1)], we know that $\omega_{\text{tor}} = \pi^* \omega_{\text{Sat}}$ is a nef line bundle on $\mathcal{A}_{g,N}^{\text{tor}}$. Similarly, since pullbacks of big line bundles along birational morphisms are big [13, Section 4.5], we know that $\omega_{\text{tor}} = \pi^* \omega_{\text{Sat}}$ is a big line bundle on $\mathcal{A}_{g,N}^{\text{tor}}$. \square

Theorem 4.4. *Suppose the characteristic of the base field \mathbb{F} is zero, or positive $\geq g(g+1)/2$ and not dividing the level $N \geq 3$. If $k \geq g+2$, then*

$$H^q \left(\mathcal{A}_{g,N}^{\text{tor}} \times \text{Spec } \mathbb{F}, (\omega_{\text{tor}})^{\otimes k}(-D) \right) = 0 \quad \text{for all } q > 0.$$

Proof. In characteristic zero, the vanishing follows from 4.3 and the Kawamata-Viehweg Vanishing Theorem 4.2, seeing as the canonical sheaf of $\mathcal{A}_{g,N}^{\text{tor}}$ is $\omega_{\text{tor}}^{\otimes g+1}(-D)$.

In positive characteristic, the vanishing theorems of Kodaira and Kawamata-Viehweg do not hold in general. However, for ω_{tor} on $\mathcal{A}_{g,N}^{\text{tor}}$, the vanishing is a special case of [14, Theorem 4.1]. \square

5 Analysis of the boundary on toroidal and Satake compactifications

Let D denote the boundary divisor of a toroidal compactification $\mathcal{A}_{g,N}^{\text{tor}}$; let Δ denote the boundary of the Satake compactification $\mathcal{A}_{g,N}^{\text{Sat}}$. It follows from [6, Theorem V.2.7] that D is the scheme-theoretic preimage of Δ under the morphism π ; in other words, the following is a fibre diagram:

$$\begin{array}{ccc} D & \xrightarrow{i} & \mathcal{A}_{g,N}^{\text{tor}} \\ \downarrow \pi|_D & & \downarrow \pi \\ \Delta & \xrightarrow{j} & \mathcal{A}_{g,N}^{\text{Sat}} \end{array}$$

The following relative vanishing result was proved independently by Andreatta-Iovita-Pilloni ([1, Proposition 8.2.2.4]) and Stroth ([22, Théorème 1]). Stroth's proof is very short and more general, as it uses fewer specific properties of Siegel modular varieties, but only works if the characteristic is at least $g(g+1)/2$. The proof in Andreatta-Iovita-Pilloni works in arbitrary characteristic, but it is based on a more intricate analysis of the behaviour of the morphism π at the boundary.

Theorem 5.1 (Andreatta-Iovita-Pilloni, Stroth). *Let \mathbb{F} be a field of arbitrary characteristic. For all $q > 0$ we have*

$$R^q \pi_* \mathcal{O}_{\mathcal{A}_{g,N}^{\text{tor}}}(-D) = 0 \quad \text{as a sheaf on } \mathcal{A}_{g,N}^{\text{Sat}} \times \text{Spec } \mathbb{F}.$$

Theorem 5.2. *If the characteristic of the base field is zero, or positive not dividing the level N , then*

$$\pi_* \mathcal{I}_D \cong \mathcal{I}_\Delta.$$

Proof. For some $m \in \mathbb{N}$, the invertible sheaf $\omega_{\text{tor}}^{\otimes m}$ is generated by its global sections [6, Proposition V.2.1]. This gives a proper morphism $\mathcal{A}_{g,N}^{\text{tor}} \rightarrow \mathbb{P}^n$ into some projective space. The Stein factorisation of this morphism (see [9, Section III.4.3]) gives

$$\begin{array}{ccc} \mathcal{A}_{g,N}^{\text{tor}} & \longrightarrow & \mathbb{P}^n \\ \downarrow \pi & \nearrow & \\ \mathcal{A}_{g,N}^{\text{Sat}} & & \end{array}$$

which defines both $\mathcal{A}_{g,N}^{\text{Sat}}$ and the proper morphism π . Consider the enlarged diagram

$$\begin{array}{ccccc} D & \xrightarrow{i} & \mathcal{A}_{g,N}^{\text{tor}} & \longrightarrow & \mathbb{P}^n \\ \downarrow \pi|_D & & \downarrow \pi & \nearrow & \\ \Delta & \xrightarrow{j} & \mathcal{A}_{g,N}^{\text{Sat}} & & \end{array}$$

where the right square is a fibre diagram. The morphism $\pi|_D$ is the base change of the proper morphism π , hence it is proper [11, Corollary II.4.8]. The two horizontal maps i and j are closed immersions; in particular i is proper [11, Corollary II.4.8] and j is finite [11, Exercise II.5.5]. Since the composition of proper morphisms is proper, and the composition of finite morphisms is finite, we can ignore most of the diagram and focus on the triangle

$$\begin{array}{ccc} D & \longrightarrow & \mathbb{P}^n \\ \downarrow \pi|_D & \nearrow & \\ \Delta & & \end{array}$$

By uniqueness, this is the Stein factorisation of the proper morphism $D \rightarrow \mathbb{P}^n$ (up to an automorphism of \mathbb{P}^n). In particular

$$(\pi|_D)_* \mathcal{O}_D \cong \mathcal{O}_\Delta.$$

Consider the defining short exact sequence for the ideal sheaf \mathcal{I}_D :

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_{\mathcal{A}_{g,N}^{\text{tor}}} \longrightarrow i_* \mathcal{O}_D \longrightarrow 0.$$

We can take higher direct images $R^\bullet \pi_*$ to get a long exact sequence of $\mathcal{O}_{\mathcal{A}_{g,N}^{\text{Sat}}}$ -modules starting with

$$0 \longrightarrow \pi_* \mathcal{I}_D \longrightarrow \pi_* \mathcal{O}_{\mathcal{A}_{g,N}^{\text{tor}}} \longrightarrow \pi_* i_* \mathcal{O}_D \longrightarrow R^1 \pi_* \mathcal{I}_D.$$

According to Theorem 5.1, the sheaf $R^1 \pi_* \mathcal{I}_D$ is zero. We get a diagram of $\mathcal{O}_{\mathcal{A}_{g,N}^{\text{Sat}}}$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_* \mathcal{I}_D & \longrightarrow & \pi_* \mathcal{O}_{\mathcal{A}_{g,N}^{\text{tor}}} & \longrightarrow & \pi_* i_* \mathcal{O}_D \longrightarrow 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathcal{I}_\Delta & \longrightarrow & \mathcal{O}_{\mathcal{A}_{g,N}^{\text{Sat}}} & \longrightarrow & j_* \mathcal{O}_\Delta \longrightarrow 0 \end{array}$$

where the middle vertical arrow is an isomorphism (by the properties of the Stein factorisation), and the right vertical arrow is an isomorphism:

$$\pi_* i_* \mathcal{O}_D = j_* ((\pi|_D)_* \mathcal{O}_D) \cong j_* \mathcal{O}_\Delta.$$

We conclude that $\pi_* \mathcal{I}_D \cong \mathcal{I}_\Delta$. □

Lemma 5.3 ([11, Exercise III.8.1]). *Suppose $\pi: X \rightarrow Y$ is a continuous map of topological spaces and \mathcal{F} is a sheaf of abelian groups on X such that*

$$R^q f_*(\mathcal{F}) = 0 \quad \text{for all } q > 0.$$

Then there are natural isomorphisms

$$H^q(X, \mathcal{F}) = H^q(Y, f_*\mathcal{F}) \quad \text{for all } q \geq 0.$$

(This is a degenerate case of the Leray spectral sequence.)

Theorem 5.4. *Suppose the characteristic of the base field \mathbb{F} is zero, or positive not dividing the level $N \geq 3$. For any $k \geq 0$ and any $q \geq 0$ we have*

$$H^q \left(\mathcal{A}_{g,N}^{\text{tor}} \times \text{Spec } \mathbb{F}, \omega_{\text{tor}}^{\otimes k}(-D) \right) = H^q \left(\mathcal{A}_{g,N}^{\text{Sat}} \times \text{Spec } \mathbb{F}, \omega_{\text{Sat}}^{\otimes k} \otimes \mathcal{I}_\Delta \right).$$

Proof. According to Theorem 5.1, we have

$$R^q \pi_* \left(\mathcal{O}_{\mathcal{A}_{g,N}^{\text{tor}}}(-D) \right) = 0 \quad \text{for all } q > 0.$$

Using the projection formula [9, Proposition 0.12.2.3], we see that

$$\begin{aligned} R^q \pi_* \left(\omega_{\text{tor}}^{\otimes k}(-D) \right) &= R^q \pi_* \left(\mathcal{O}_{\mathcal{A}_{g,N}^{\text{tor}}}(-D) \otimes \pi^* \omega_{\text{Sat}}^{\otimes k} \right) \\ &= R^q \pi_* \left(\mathcal{O}_{\mathcal{A}_{g,N}^{\text{tor}}}(-D) \right) \otimes \omega_{\text{Sat}}^{\otimes k} \\ &= 0. \end{aligned}$$

We conclude that

$$H^q \left(\mathcal{A}_{g,N}^{\text{tor}}, \omega_{\text{tor}}^{\otimes k}(-D) \right) = H^q \left(\mathcal{A}_{g,N}^{\text{Sat}}, \pi_* \left(\omega_{\text{tor}}^{\otimes k}(-D) \right) \right) = H^q \left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k} \otimes \mathcal{I}_\Delta \right),$$

where the first equality comes from Lemma 5.3 and the second equality from Theorem 5.2. \square

The following result² is well-known as part of the proof of Grothendieck's cohomological dimension theorem, see the original [8, Théorème 3.6.5] or the presentation in [11, Theorem III.2.7]:

Lemma 5.5. *Let X be a topological space with finitely many irreducible components. Let \mathcal{F} be a sheaf of abelian groups on X , and let $q \geq 0$. If $H^q(Y, \mathcal{F}|_Y) = 0$ for all irreducible components Y of X , then $H^q(X, \mathcal{F}) = 0$.*

Theorem 5.6. *Suppose the characteristic of the base field \mathbb{F} is zero, or positive $\geq g(g+1)/2$ and not dividing the level $N \geq 3$. For all $k \geq g+2$ and $q > 0$, we have*

$$H^q \left(\mathcal{A}_{g,N}^{\text{Sat}} \times \text{Spec } \mathbb{F}, \omega_{\text{Sat}}^{\otimes k} \right) = 0.$$

²Grothendieck attributes to Serre this *astuce* of reducing to the case of an irreducible space, see [10, page 29].

Proof. We base change our spaces to \mathbb{F} and omit $\text{Spec } \mathbb{F}$ from the notation, for simplicity.

We proceed by induction on g .

The base case $g = 1$ is well-known but worth including. Here $\mathcal{A}_{g,N}^{\text{Sat}} = \mathcal{A}_{g,N}^{\text{tor}}$ is the modular curve $X(N)$, so the vanishing is clear for $q > 1$. The sheaf ω_{Sat} has positive degree, and so does the effective divisor D . By Serre duality, we have

$$H^1\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}\right) = H^0\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes 2-k}(-D)\right)^{\vee}.$$

But if $k \geq g + 2 = 3$, then $2 - k < 0$ so $\omega_{\text{Sat}}^{\otimes 2-k}(-D)$ has negative degree, and hence no nonzero global sections.

For the induction step, consider the short exact sequence that defines the ideal sheaf \mathcal{I}_{Δ} :

$$0 \longrightarrow \mathcal{I}_{\Delta} \longrightarrow \mathcal{O}_{\mathcal{A}_{g,N}^{\text{Sat}}} \longrightarrow j_* \mathcal{O}_{\Delta} \longrightarrow 0.$$

Tensoring with the line bundle $\omega_{\text{Sat}}^{\otimes k}$ gives another short exact sequence

$$0 \longrightarrow \mathcal{I}_{\Delta} \otimes \omega_{\text{Sat}}^{\otimes k} \longrightarrow \omega_{\text{Sat}}^{\otimes k} \longrightarrow \omega_{\text{Sat}}^{\otimes k}|_{\Delta} \longrightarrow 0. \quad (5.1)$$

Applying Theorems 5.4, then 4.4, for $k \geq g + 2$ we have

$$H^q\left(\mathcal{A}_{g,N}^{\text{Sat}}, \mathcal{I}_{\Delta} \otimes \omega_{\text{Sat}}^{\otimes k}\right) = H^q\left(\mathcal{A}_{g,N}^{\text{tor}}, \mathcal{I}_D \otimes \omega_{\text{tor}}^{\otimes k}\right) = 0.$$

The long exact sequence of cohomology associated with (5.1) has pieces of the form

$$H^q\left(\mathcal{A}_{g,N}^{\text{Sat}}, \mathcal{I}_{\Delta} \otimes \omega_{\text{Sat}}^{\otimes k}\right) \longrightarrow H^q\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}\right) \longrightarrow H^q\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}|_{\Delta}\right).$$

If $q > 0$, we have just seen that the leftmost group is zero, so it suffices to prove that the rightmost group is zero.

Let C_1, \dots, C_b be the irreducible components of the boundary Δ . Each component C_j is isomorphic to $\mathcal{A}_{g-1,N}^{\text{Sat}}$, with

$$\omega_{\text{Sat},g}|_{C_j} = \omega_{\text{Sat},g-1}$$

(by [6, Theorem V.2.5(4)]). By the induction hypothesis, $H^q(\mathcal{A}_{g-1,N}^{\text{Sat}}, \omega_{\text{Sat},g-1}^{\otimes k}) = 0$. So each component C_j of Δ satisfies

$$H^q(C_j, \omega_{\text{Sat}}^{\otimes k}|_{C_j}) = 0.$$

Finally, Lemma 5.5 allows us to conclude that $H^q(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}|_{\Delta}) = 0$. \square

Corollary 5.7. *Let $N \geq 3$. Suppose $p \geq g(g+1)/2$ is a prime not dividing N . For all $k \geq g + 2$, the base change morphism*

$$M_k(\Gamma(N)) \otimes \mathbb{F}_p \longrightarrow M_k(\Gamma(N); \mathbb{F}_p)$$

is an isomorphism.

Proof. Over the local Noetherian ring \mathbb{Z}_p , Theorem 5.6 and Corollary 3.2 imply that the base change morphism is an isomorphism. By flat base change this implies that

$$H^0\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}\right) \otimes_{\mathbb{Z}[1/N]} \mathbb{F}_p \longrightarrow H^0\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k} \otimes_{\mathbb{Z}[1/N]} \mathbb{F}_p\right)$$

is an isomorphism. The result now follows by Köcher's principle. \square

See Corollary 8.2 for an extension of this result to the small levels $N = 1, 2$.

6 Vanishing of vector bundles in characteristic zero

We start with a variant of a vanishing theorem for cohomology of vector bundles, due to Demailly. We follow Manivel's simplified proof of this result, as presented in [16, Section 7.3.B].

Given a vector bundle E on a projective scheme X , let $\mathbb{P}(E)$ denote the projective bundle of E parametrising hyperplane sections in the fibres E_x . We say that E is *nef* over X if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle over $\mathbb{P}(E)$. We refer the reader to [16, Section 6.2.B] for basic properties of nef vector bundles, and to [15, Appendix A] for a short summary of projective bundles.

Theorem 6.1 (Demailly-Manivel). *Let X be a smooth projective complex variety.*

Let E be a nef vector bundle of rank e on X , and let L be a nef and big line bundle on X .

Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_e) \in \mathbb{Z}^e$, $\lambda_e \geq 0$; let $h = h(\lambda)$ denote the number of nonzero parts λ_i of λ , and let E^λ be the vector bundle associated to the irreducible representation of GL_e with highest weight λ .

Then

$$H^q(X, \omega_X \otimes E^\lambda \otimes (\det E)^{\otimes h} \otimes L) = 0 \quad \text{for all } q > 0.$$

Proof. By the definition of h we have $\lambda = (\lambda_1 \geq \dots \geq \lambda_h \geq 0 \geq \dots \geq 0)$. Let $m = \lambda_1 + \dots + \lambda_h$, and let $F = E \oplus E \oplus \dots \oplus E$, where we take h summands. Then F is a nef vector bundle on X , and $\det F = (\det E)^{\otimes h}$.

We apply Theorem 6.2 to F and get

$$H^q(X, \omega_X \otimes (\mathrm{Sym}^m F) \otimes (\det F) \otimes L) = 0 \quad \text{for all } q > 0. \quad (6.1)$$

But

$$\mathrm{Sym}^m F = \bigoplus_{m_1 + \dots + m_h = m} (\mathrm{Sym}^{m_1} E) \otimes \dots \otimes (\mathrm{Sym}^{m_h} E)$$

In particular, we have

$$E^\lambda \subset (\mathrm{Sym}^{\lambda_1} E) \otimes \dots \otimes (\mathrm{Sym}^{\lambda_h} E) \subset \mathrm{Sym}^m F,$$

where both inclusions are as direct summands. Therefore (6.1) gives us

$$H^q(X, \omega_X \otimes E^\lambda \otimes (\det E)^{\otimes h} \otimes L) = 0 \quad \text{for all } q > 0.$$

□

For completeness, we give a proof of the following variant of the Griffiths vanishing theorem, which is stated in [16, Example 7.3.3].

Theorem 6.2 (Griffiths). *Let X be a smooth projective complex variety of dimension n .*

Let F be a nef vector bundle of rank r on X , and let L be a nef and big line bundle on X . Then

$$H^q(X, \omega_X \otimes (\mathrm{Sym}^m F) \otimes (\det F) \otimes L) = 0 \quad \text{for all } q > 0, m \geq 0.$$

Proof. The cotangent bundle sequence for $\pi: \mathbb{P}(F) \rightarrow X$ gives $\omega_{\mathbb{P}(F)} = \omega_{\mathbb{P}(F)/X} \otimes \pi^* \omega_X$. The relative Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}(F)/X}^1 \longrightarrow \pi^* F \otimes \mathcal{O}_{\mathbb{P}(F)}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(F)} \longrightarrow 0$$

gives $\omega_{\mathbb{P}(F)/X} = \pi^* \det F \otimes \mathcal{O}_{\mathbb{P}(F)}(-r)$. Then for any $m \geq 0$,

$$\begin{aligned} \pi_*(\omega_{\mathbb{P}(F)} \otimes \mathcal{O}_{\mathbb{P}(F)}(m+r)) &= \pi_*(\pi^*(\omega_X \otimes \det F) \otimes \mathcal{O}_{\mathbb{P}(F)}(m)) \\ &= \omega_X \otimes \det F \otimes \pi_*(\mathcal{O}_{\mathbb{P}(F)}(m)) \\ &= \omega_X \otimes \det F \otimes \text{Sym}^m F. \end{aligned}$$

Furthermore, when $m \geq 0$ we have

$$R^{r-1}\pi_*\mathcal{O}_{\mathbb{P}(F)}(-r-m) = (\text{Sym}^m F)^* \otimes \det F^*$$

and all other direct image sheaves of this type vanish. By the projection formula

$$\begin{aligned} R^i\pi_*(\omega_{\mathbb{P}(F)} \otimes \mathcal{O}_{\mathbb{P}(F)}(m+r) \otimes \pi^*L) &= R^i\pi_*(\mathcal{O}_{\mathbb{P}(F)}(m) \otimes \pi^*(\omega_X \otimes \det F \otimes L)) \\ &= R^i\pi_*(\mathcal{O}_{\mathbb{P}(F)}(m)) \otimes \omega_X \otimes \det F \otimes L, \end{aligned}$$

so the higher direct image sheaves of $\omega_{\mathbb{P}(F)} \otimes \mathcal{O}_{\mathbb{P}(F)}(m+r) \otimes \pi^*L$ vanish. Therefore

$$H^i(X, \omega_X \otimes \det F \otimes \text{Sym}^m F \otimes L) = H^i(\mathbb{P}(F), \omega_{\mathbb{P}(F)} \otimes \mathcal{O}_{\mathbb{P}(F)}(m+r) \otimes \pi^*L).$$

Observe that $\mathcal{O}_{\mathbb{P}(F)}(1)$ is nef (by definition) and π^*L is also nef (as the pullback of a nef line bundle under a proper, surjective map). Hence $\mathcal{O}_{\mathbb{P}(F)}(m+r) \otimes \pi^*L$ is nef.

To show that $\mathcal{O}_{\mathbb{P}(F)}(m+r) \otimes \pi^*L$ is big we use the fact that a nef divisor is big if and only if its top intersection is strictly positive [15, Theorem 2.2.16]. Since the sum of a nef divisor and a nef and big divisor is nef and big (a nef divisor that is not big lies on an extremal ray of the nef cone) it suffices to show that $\mathcal{O}_{\mathbb{P}(F)}(1) \otimes \pi^*L$ is nef and big.

We have

$$(c_1(\mathcal{O}_{\mathbb{P}(F)}(1)) + c_1(\pi^*L))^{n+r-1} = \sum_{i=0}^{n+r-1} \binom{n+r-1}{i} c_1(\mathcal{O}_{\mathbb{P}(F)}(1))^i \cdot c_1(\pi^*L)^{n+r-1-i}.$$

An ample divisor restricted to any subvariety is ample. As a result of this in our situation we have $A_1 \cdots A_{n+r-1} > 0$ for ample A_i . In the limit this gives $D_1 \cdots D_{n+r-1} \geq 0$ for nef D_i . Hence $c_1(\mathcal{O}_{\mathbb{P}(F)}(1))^i \cdot c_1(\pi^*L)^{n+r-1-i} \geq 0$.

It remains to exhibit one non-zero term in the sum. We know that $\deg(c_1(L)^n) > 0$ as L is nef and big. The fibres of $\mathbb{P}(F) \rightarrow X$ are isomorphic to \mathbb{P}^{r-1} . Now $\mathcal{O}_{\mathbb{P}(F)}(1)$ restricted to each fibre is $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ hence $c_1(\pi^*L)^n \cdot c_1(\mathcal{O}_{\mathbb{P}(F)}(1))^{r-1} = \deg(c_1(L)^n) > 0$.

So we have that $\mathcal{O}_{\mathbb{P}(F)}(m+r) \otimes \pi^*L$ is nef and big and an application of Kawamata-Viehweg vanishing gives the result we require. \square

Our interest in these vanishing results comes from the fact that the vector bundle \mathbb{E}_{tor} on $\mathcal{A}_{g,N}^{\text{tor}}$ is nef. (See the proof of [18, Corollary 3.2], where this fact is credited to Kawamata. Beware that \mathbb{E}_{tor} is denoted $F^{1,0}$ in [18].)

To simplify the statement of some of the following results, we define what we mean by an element of \mathbb{Z}^g to be “sufficiently large” with respect to g . Let

$$\mu = (\mu_1 + k \geq \dots \geq \mu_{g-1} + k \geq k) \in \mathbb{Z}^g \quad \text{where } \mu_{g-1} \geq 0.$$

Let $\lambda = (\mu_1 \geq \dots \geq \mu_{g-1} \geq 0)$ and let $h(\lambda)$ be the number of nonzero μ_i 's. If $k \geq g + h(\lambda) + 2$, we say that μ is “sufficiently large”.

Theorem 6.3. *Let $g \geq 2$. If $\mu \in \mathbb{Z}^g$ is “sufficiently large”, then*

$$H^q(\mathcal{A}_{g,N}^{\text{tor}}, \mathbb{E}_{\text{tor}}^\mu(-D)) = 0 \quad \text{for all } q > 0.$$

Proof. The canonical bundle of $X = \mathcal{A}_{g,N}^{\text{tor}}$ is [6, Section VI.4]

$$\omega_X = (\omega_{\text{tor}})^{\otimes g+1}(-D).$$

Let $j = k - g - h(\lambda) - 1 > 0$. Note that $\mathbb{E}_{\text{tor}}^\mu = \mathbb{E}_{\text{tor}}^\lambda \otimes (\omega_{\text{tor}})^{\otimes k}$. We apply Theorem 6.1 with $E = \mathbb{E}_{\text{tor}}$ and $L = (\omega_{\text{tor}})^{\otimes j}$ and get

$$\begin{aligned} H^q(\mathcal{A}_{g,N}^{\text{tor}}, \mathbb{E}_{\text{tor}}^\mu(-D)) &= H^q(\mathcal{A}_{g,N}^{\text{tor}}, \mathbb{E}_{\text{tor}}^\lambda \otimes (\omega_{\text{tor}})^{\otimes k}(-D)) \\ &= H^q(\mathcal{A}_{g,N}^{\text{tor}}, (\omega_{\text{tor}})^{\otimes g+1}(-D) \otimes \mathbb{E}_{\text{tor}}^\lambda \otimes (\omega_{\text{tor}})^{\otimes h(\lambda)} \otimes (\omega_{\text{tor}})^{\otimes j}) \\ &= 0. \end{aligned}$$

□

We record two special cases of interest. First, note that the case of highest weight $\mu = (k \geq \dots \geq k) \in \mathbb{Z}^g$ gives precisely the vanishing result for scalar-valued forms which constitutes the characteristic zero part of Theorem 4.4. The second special case is that of symmetric powers, corresponding to highest weight $\mu = (j + k \geq k \geq \dots \geq k) \in \mathbb{Z}^g$:

Corollary 6.4 (Symmetric powers). *If $j \geq 1$ and $k \geq g + 3$, then*

$$H^q(\mathcal{A}_{g,N}^{\text{tor}}, \text{Sym}^j(\mathbb{E}_{\text{tor}}) \otimes (\omega_{\text{tor}})^{\otimes k}(-D)) = 0 \quad \text{for all } q > 0.$$

7 Surjectivity of the Siegel operators

Let ρ be an irreducible representation of the algebraic group GL_g with highest weight vector $\lambda = (\lambda_1, \dots, \lambda_g)$. If $\Delta = \mathcal{A}_g^{\text{Sat}} - \mathcal{A}_g$ then the inclusion $i: \Delta \hookrightarrow \mathcal{A}_g^{\text{Sat}}$ gives

$$H^0(\mathcal{A}_g^{\text{Sat}}, \mathbb{E}_{\text{Sat}}^\rho \otimes i_* \mathcal{O}_\Delta \otimes B) = H^0(\mathcal{A}_{g-1}^{\text{Sat}}, \mathbb{E}_{\text{Sat}}^{\rho'} \otimes B),$$

where ρ' is the irreducible representation of GL_{g-1} with highest weight vector $\lambda' = (\lambda_1, \dots, \lambda_{g-1})$.

The map that takes a section of $H^0(\mathcal{A}_g^{\text{Sat}}, \mathbb{E}_{\text{Sat}}^\rho \otimes B)$ to its restriction as a section in $H^0(\mathcal{A}_{g-1}^{\text{Sat}}, \mathbb{E}_{\text{Sat}}^{\rho'} \otimes B)$ is known as the Siegel Φ -operator. When $B = \mathbb{C}$, this operator

$$\Phi: M_\rho^g(1, \mathbb{C}) \longrightarrow M_{\rho'}^{g-1}(1, \mathbb{C})$$

is realised as

$$(\Phi f)(\tau') = \lim_{t \rightarrow \infty} f \begin{pmatrix} it & 0 \\ 0 & \tau' \end{pmatrix}$$

for $\tau' \in \mathcal{H}_{g-1}$, $t \in \mathbb{R}$.

Weissauer shows [24, Korollar zum Satz 8, p. 87] that Φ is surjective for even $k \geq g+2$. In the vector-valued case with $g = 2$, $\mu = (j+k, k)$ it was proved by Arakawa to be surjective for $j \geq 1$ and $k \geq 5$ in [2, Proposition 1.3]. Weissauer and Arakawa's proofs are analytic in nature and involve showing certain integrals representing an averaging process converge.

The Siegel Φ -operator generalises to higher levels as the restriction of global sections to the boundary of the Siegel variety. If $\Delta = \mathcal{A}_{g,N}^{\text{Sat}} - \mathcal{A}_{g,N}$ denotes the boundary of the Satake compactification, then the inclusion $i: \Delta \hookrightarrow \mathcal{A}_{g,N}^{\text{Sat}}$ gives the operator

$$\Phi_{\text{Sat}}: H^0(\mathcal{A}_{g,N}^{\text{Sat}}, \mathbb{E}_{\text{Sat}}^\rho \otimes B) \longrightarrow H^0(\mathcal{A}_{g,N}^{\text{Sat}}, \mathbb{E}_{\text{Sat}}^\rho \otimes i_* \mathcal{O}_\Delta \otimes B).$$

Similarly, if $D = \mathcal{A}_{g,N}^{\text{tor}} - \mathcal{A}_{g,N}$ denotes the boundary divisor of the toroidal compactification, then the inclusion $j: D \hookrightarrow \mathcal{A}_{g,N}^{\text{tor}}$ gives the operator

$$\Phi_{\text{tor}}: H^0(\mathcal{A}_{g,N}^{\text{tor}}, \mathbb{E}_{\text{tor}}^\rho \otimes B) \longrightarrow H^0(\mathcal{A}_{g,N}^{\text{tor}}, \mathbb{E}_{\text{tor}}^\rho \otimes j_* \mathcal{O}_D \otimes B).$$

We investigate conditions under which these operators are surjective.

Consider the ideal sheaf of $j: D \hookrightarrow \mathcal{A}_{g,N}^{\text{tor}}$, defined by the short exact sequence

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_{\mathcal{A}_{g,N}^{\text{tor}}} \longrightarrow j_* \mathcal{O}_D \longrightarrow 0.$$

Since $\mathbb{E}_{\text{tor}}^\rho$ is locally free, tensoring by $\mathbb{E}_{\text{tor}}^\rho$ gives a short exact sequence

$$0 \longrightarrow \mathbb{E}_{\text{tor}}^\rho \otimes \mathcal{I}_D \longrightarrow \mathbb{E}_{\text{tor}}^\rho \longrightarrow \mathbb{E}_{\text{tor}}^\rho|_D \longrightarrow 0. \quad (7.1)$$

We get a long exact sequence in cohomology that features the toroidal operator

$$0 \longrightarrow S_\mu(N) \longrightarrow M_\mu(N) \xrightarrow{\Phi_{\text{tor}}} H^0(D, \mathbb{E}_{\text{tor}}^\mu|_D) \longrightarrow H^1(\mathcal{A}_{g,N}^{\text{tor}}, \mathbb{E}_{\text{tor}}^\mu \otimes \mathcal{I}_D).$$

The following is a direct consequence of Theorem 6.3. (For the definition of “sufficiently large”, see the paragraph before Theorem 6.3.)

Theorem 7.1. *Let $N \geq 3$. Over a field of characteristic zero, we have*

- (i) *If μ is “sufficiently large”, then Φ_{tor} on forms of degree g and weight μ is surjective.*
- (ii) *If $k \geq g + 2$, then Φ_{tor} on scalar-valued forms of degree g and weight k is surjective.*
- (iii) *If $j \geq 1$ and $k \geq g + 3$, then Φ_{tor} on forms of degree g and weight $\text{Sym}^j \otimes \det^{\otimes k}$ is surjective.*

Note that part (iii) is the toroidal analogue in level $N \geq 3$ of a result proved by Arakawa in degree 2 and level $N = 1$ for the Satake compactification, see [2, Proposition 1.3].

In positive characteristic, we can restrict to scalar-valued forms and appeal to the vanishing theorem of Lan and Suh (Theorem 4.4) to get:

Theorem 7.2. *Let $p \geq g(g+1)/2$ be a prime not dividing the level $N \geq 3$. If $k \geq g + 2$, then Φ_{tor} on scalar-valued forms (mod p) of degree g and weight k is surjective.*

The operator Φ_{Sat} also fits into a long exact sequence

$$0 \longrightarrow S_k(N) \longrightarrow M_k(N) \xrightarrow{\Phi_{\text{Sat}}} H^0(\Delta, \omega_{\text{Sat}}^{\otimes k}|_\Delta) \longrightarrow H^1(\mathcal{A}_{g,N}^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k} \otimes \mathcal{I}_\Delta).$$

By appealing to Theorems 7.1(ii), 7.2 and 5.4, we obtain

Corollary 7.3. *Suppose the characteristic of the base field \mathbb{F} is zero, or a prime $p \geq g(g+1)/2$ not dividing the level $N \geq 3$. If $k \geq g + 2$, then Φ_{Sat} on scalar-valued forms over \mathbb{F} of degree g and weight k is surjective.*

In characteristic zero, this gives an algebraic proof for a result analogous to [24, Korollar zum Satz 8, p. 87], which was obtained by Weissauer using analytic methods. See Theorem 8.4 for a version of Theorem 7.2 and Corollary 7.3 in level 1, and Theorem 8.5 for a result about lifting forms of level 1 to forms of level $N \geq 3$ in higher degree.

8 Levels 1 and 2

If $N \geq 3$, then for any $L \geq 1$ the canonical morphism

$$\mathcal{A}_{g, LN} \longrightarrow \mathcal{A}_{g, N}$$

is a finite covering with Galois group $G(g, L) := \mathrm{GSp}_{2g}(\mathbb{Z}/L\mathbb{Z})$. Therefore, given any $\mathbb{Z}[1/LN]$ -algebra B , we have

$$\begin{aligned} M_\rho(\Gamma(N); B) &= M_\rho(\Gamma(LN); B)^{G(g, L)} \\ S_\rho(\Gamma(N); B) &= S_\rho(\Gamma(LN); B)^{G(g, L)} \end{aligned}$$

We use these formulas to *define* Siegel modular forms and cusp forms of levels $N = 1$ and 2 and coefficients in B . (This is independent of the choice of L invertible in B .)

If p is a prime not dividing the order of $G(g, L)$, then the “invariants” functor $\mathbb{Z}_p[G(g, L)]\text{-}\mathbf{Mod} \rightarrow \mathbb{Z}_p\text{-}\mathbf{Mod}$ given by $M \mapsto M^{G(g, L)}$ is exact. By the Chinese Remainder Theorem,

$$\#G(g, L) = \prod_{\ell^a \parallel L, \ell \text{ prime}} \#G(g, \ell^a).$$

On the other hand, it is known that if ℓ is prime

$$\#G(g, \ell^a) = (\ell^a - 1) \ell^{ag^2} \prod_{i=1}^g (\ell^{2ia} - 1),$$

which in particular shows that $\#G(g, \ell)$ divides $\#G(g, \ell^a)$. We conclude that, in order to find L such that a particular prime p does not divide $\#G(g, L)$, it is sufficient to consider prime numbers $\ell = L$.

Proposition 8.1. *Let $g \geq 1$ be an integer. Let p be a prime $> 2g + 1$. There exists a prime number $\ell \geq 3$ such that p does not divide*

$$\#G(g, \ell) = (\ell - 1) \ell^{g^2} \prod_{i=1}^g (\ell^{2i} - 1).$$

Moreover, the inequality $p > 2g + 1$ is sharp.

Proof. Let $\ell \geq 3$ be a prime primitive root modulo p . (It is known that there are infinitely many such ℓ , see for instance [17].)

Then the order of ℓ in the group $(\mathbb{Z}/p\mathbb{Z})^\times$ is exactly $p - 1 > 2g$, in other words $\ell^{2g} \not\equiv 1 \pmod{p}$, so $p \nmid (\ell^{2g} - 1)$. The same argument forbids p from dividing any of the factors in $\#G(g, \ell)$.

The claim about the sharpness of the inequality $p > 2g + 1$ can be stated more precisely as follows: if $g \geq 1$ and p is a prime $\leq 2g + 1$, then p divides $\#G(g, \ell)$ for all primes $\ell \geq 3$. This is easily checked for $g = 1$. For general g , the formula for $\#G(g, \ell)$ shows that

$$\#G(g, \ell) = \#G(g - 1, \ell) \cdot \ell^{2g-1} \cdot (\ell^{2g} - 1).$$

By Fermat’s little theorem, if $p = 2g + 1$ is prime, then either $\ell = p$ or p divides $\ell^{2g} - 1$. A simple induction argument concludes the proof. \square

We summarize the content of this section and its relevance to the rest of the paper:

Corollary 8.2. *Let $g \geq 1$ and p be a prime $> 2g + 1$. There exists a prime number $\ell \geq 3$ such that the functor $\mathbb{Z}_p[G(g, \ell)]\text{-Mod} \rightarrow \mathbb{Z}_p\text{-Mod}$, $M \mapsto M^{G(g, \ell)}$, is exact.*

In particular, the statement of Corollary 5.7 can be extended as follows: let $g \geq 1$ and p be a prime $> 2g + 1$ not dividing N . For all $k \geq g + 2$ the base change morphisms

$$\begin{aligned} M_k(\Gamma(N)) \otimes \mathbb{F}_p &\longrightarrow M_k(\Gamma(N); \mathbb{F}_p) \\ S_k(\Gamma(N)) \otimes \mathbb{F}_p &\longrightarrow S_k(\Gamma(N); \mathbb{F}_p) \end{aligned}$$

are isomorphisms.

Remark 8.3. The alert reader will have noticed a gap between the condition $p \geq (g+1)g/2$ from Corollary 5.7 and the condition $p > 2g + 1$ assumed in Corollary 8.2. More precisely, the cases:

- (a) $g = 1, p = 2, 3$
- (b) $g = 2, p = 3, 5$
- (c) $g = 3, p = 7$

are not covered by Corollary 8.2. For (a), which is classical, see [4, Lemma 1.9]. Cases (b) and (c) can presumably be studied in a similar way, using explicit presentations over \mathbb{Z} of the ring of scalar-valued Siegel modular forms of degree 2, respectively 3.

It is natural to ask whether small level versions of Theorem 7.2 and Corollary 7.3 also follow from the result in Corollary 8.2. This is however not the case, at least not directly, since the Siegel Φ -operator involves spaces with actions of different groups. In level 1, we can deduce the surjectivity of the Siegel operator in positive characteristic from Weissauer's result over \mathbb{C} :

Theorem 8.4. *Let $g \geq 2$ and $k \geq g + 2$ and even, and \mathbb{F} be a field of characteristic zero or $p > 2g + 1$. Then the natural map*

$$M_k^g(1; \mathbb{F}) = H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k} \otimes \mathbb{F}\right) \xrightarrow{\Phi = \Phi_{\text{Sat}}} H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}|_{\Delta} \otimes \mathbb{F}\right) = M_k^{g-1}(1; \mathbb{F})$$

is surjective.

Proof. Weissauer proved this result over the field \mathbb{C} in [24, Korollar zum Satz 8, p. 87]. Through the flat base change $\mathbb{Z} \hookrightarrow \mathbb{C}$ we know that this implies

$$H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}\right) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\Phi \otimes \text{id}} H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}|_{\Delta}\right) \otimes_{\mathbb{Z}} \mathbb{C}$$

is surjective. Hence we know that

$$M_k^g(1; \mathbb{Z}) = H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}\right) \xrightarrow{\Phi} H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}|_{\Delta}\right) = M_k^{g-1}(1; \mathbb{Z})$$

is a \mathbb{Z} -linear map of full rank. Suppose it is not surjective, i.e. there is some $f \in M_k^{g-1}(1; \mathbb{Z})$ such that f is not in the image of Φ . Since the map has full rank, its cokernel is torsion, so there is a minimal $m \in \mathbb{N}$ such that mf is in the image of Φ . If the field \mathbb{F} has characteristic zero, then $m^{-1} \in \mathbb{F}$ and Φ is surjective.

It remains to deal with the case where \mathbb{F} has characteristic $p > 2g + 1$. Let $F \in M_k^g(1; \mathbb{Z})$ be such that $\Phi(F) = mf$ for $m \in \mathbb{N}$ minimal, and let $\overline{F} \in M_k^g(1; \mathbb{F}_p)$ be the reduction of F modulo p .

Suppose that $p \mid m$. Then $\Phi(\overline{F}) = m\overline{f} = 0$, so $\overline{F} \in \ker \Phi = S_k^g(1; \mathbb{F}_p)$. Since $p > 2g + 1$, by Corollary 8.2 we know that \overline{F} must lift to a cusp form $G \in S_k^g(1; \mathbb{Z})$. Then $(F - G)(q) \equiv 0 \pmod{p}$, so all the Fourier coefficients of $F - G$ are divisible by p , therefore $\frac{1}{p}(F - G) \in M_k^g(1; \mathbb{Z})$. This gives

$$\Phi\left(\frac{1}{p}(F - G)\right) = \frac{m}{p}f,$$

contradicting the minimality of m .

We conclude that p does not divide m . Hence the map

$$H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}\right) \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\Phi \otimes \text{id}} H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega_{\text{Sat}}^{\otimes k}|_{\Delta}\right) \otimes_{\mathbb{Z}} \mathbb{F}_p$$

is surjective for $p > 2g + 1$ and Corollary 8.2 gives the result for \mathbb{F}_p . Flat base change $\mathbb{F}_p \hookrightarrow \mathbb{F}$ extends it to other fields \mathbb{F} of characteristic $p > 2g + 1$. \square

Note that the condition that k be even (and not just $\geq g + 2$) is necessary, even over \mathbb{C} : there is a cusp form χ of level 1, degree 2 and weight $35 \geq 3 + 2$, but χ is not in the image of Φ since there are no forms of level 1, degree 3 and odd weight. In fact, if kg is odd then $M_k^g(N; \mathbb{Z}) = 0$ for $N = 1, 2$ as $-I \in \Gamma(N)$ implies $f = -f$. However, our results do give some insight into behaviour in level 1 for odd weights.

Theorem 8.5. *Suppose the characteristic of the base field \mathbb{F} is zero, or a prime $p \geq g(g + 1)/2$ not dividing the level $N \geq 3$. If $k \geq g + 2$, there is a commutative diagram*

$$\begin{array}{ccc} H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega^{\otimes k}\right) & \hookrightarrow & H^0\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega^{\otimes k}\right) \cong M_k^g(N; \mathbb{F}) \\ \downarrow \Phi & & \downarrow \Phi_{\text{Sat}} \\ M_k^{g-1}(1; \mathbb{F}) \cong H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega^{\otimes k}|_{\Delta_1}\right) & \xrightarrow{\Psi} & H^0\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega^{\otimes k}|_{\Delta_N}\right) \end{array}$$

such that for any $f \in M_k^{g-1}(1; \mathbb{F})$ there exists an $F \in M_k^g(N; \mathbb{F})$ with $\Phi_{\text{Sat}}(F) = \Psi(f)$.

Proof. Write $\Delta_N = \mathcal{A}_{g,N}^{\text{Sat}} - \mathcal{A}_{g,N}$ for the boundary of the level N Satake compactification. Since the covering morphism $\pi: \mathcal{A}_{g,N}^{\text{Sat}} \rightarrow \mathcal{A}_g^{\text{Sat}}$ is finite, we have $\mathcal{O}_{\Delta_1} = \pi_* \mathcal{O}_{\Delta_N}$. We also know that $\pi^* \omega = \omega$, so by the projection formula

$$\omega^{\otimes k}|_{\Delta_1} = \omega^{\otimes k} \otimes \pi_* \mathcal{O}_{\Delta_N} = \pi_* \left((\pi^* \omega^{\otimes k}) \otimes \mathcal{O}_{\Delta_N} \right) = \pi_* \omega^{\otimes k}|_{\Delta_N}.$$

This means that we have an injection of global sections

$$\Psi: M_k^{g-1}(1; \mathbb{F}) \cong H^0\left(\mathcal{A}_g^{\text{Sat}}, \omega^{\otimes k}|_{\Delta_1}\right) \hookrightarrow H^0\left(\mathcal{A}_{g,N}^{\text{Sat}}, \omega^{\otimes k}|_{\Delta_N}\right)$$

which is the missing feature in the commutative diagram in the statement.

Corollary 7.3 gives conditions for Φ_{Sat} to be surjective. Under these conditions we have that any element embedded by the above map will have a pre-image under Φ_{Sat} in $M_k^g(N; \mathbb{F})$. \square

Over \mathbb{C} we consider the restriction to just one of the irreducible components of the cusp of the Satake compactification to highlight the relevance of this algebraic result in the more analytic setting of the results of Weissauer [24, Korollar zum Satz 8, p. 87].

Corollary 8.6. *Let $k \geq g + 2$, $N \geq 3$. Then for any $f \in M_k^{g-1}(1; \mathbb{C})$ there exists an $F \in M_k^g(N; \mathbb{C})$ such that*

$$\lim_{t \rightarrow \infty} F \begin{pmatrix} it & 0 \\ 0 & \tau \end{pmatrix} = f(\tau)$$

for $\tau \in \mathcal{H}_{g-1}$, $t \in \mathbb{R}$.

(Weissauer's result implies this for *even* weights as any level 1 form can be considered a form of level $N \geq 1$. In comparison, Corollary 8.6 applies in *both even and odd* weights.)

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